# Estimation of the Effect of Multiplicative Noise on Signal Resolution by the Woodward Criterion

Artyushenko Vladimir Mikhaylovich
Information technology and management systems department
Technological university
Korolev city, Russian Federation
artuschenko@mail.ru

Volovach Vladimir Ivanovich
Informational and electronic service department
Volga region state university of service
Togliatty city, Russian Federation
volovach.vi@mail.ru

***Abstract*—Issues related to estimating the influence of quasi-deterministic and fluctuating multiplicative noise on the delay resolution and frequency resolution of systems processing radio signals based on the Woodward criterion for narrow-band and broadband signals are considered. The problem of resolution when detecting or measuring signal parameters can be considered both for useful signals alone and for cases when interfering signals are present as well. It is pointed out that the effect of multiplicative noise on the signal almost always leads to a resolution problem. It is shown, that under the influence of significant broadband multiplicative noise, the time resolution interval is determined only by the signal envelope and does not depend on its phase structure. For instance, for signals with a rectangular or bell-shaped envelope, it is equal to the equivalent signal duration. Examples of calculating resolution intervals under the influence of multiplicative noise are given. Taking into account the effect of multiplicative noise on signal resolution leads to an increase in the efficiency of radio systems, detection of objects being an example.**

***Keywords—radio engineering system, resolution problem, resolution, Woodward criterion, noise modulation function***

## I. INTRODUCTION

For a wide class of radio systems, along with the primary characteristics that determine the quality of their operation, such as the probability of detecting a signal and the accuracy of measuring its parameters, there are indicators determining the ability to separately detect or measure parameters of signals with closely located responses at the output of the system's receiver. The problem of resolution of separate observations or measurement of signals' parameters occurs, for example, in radio detection and location when two or more closely located targets are observed. In this case, signals from all targets are useful. Other radio systems may have the problem of resolution of multiple signals, when one of them is useful while all others are interfering. Interfering signals can be generated by other radio devices of the same type and by systems operating in close proximity to the radio system in question.

Note that in many cases, although in the absence of multiplicative (modulating) noise (MN), problems of resolution do not arise, since the mutual influence of signals on each other and on the operation of the system for each signal separately is insignificant, in the presence of MN such mutual influence can be very noticeable. This is due to the fact, that in signals distorted by MN, a noise component appears, which can create the output effect of the receiver at the values of the signal parameters, such as arrival time and frequency shift, whereas the output effect on them in the absence of MN is negligible.

The effect of MN on the resolution of systems whose linear part of the receiving device includes a filter matched with the undistorted signal is also due to a decrease in the output signal power relative to the power of additive noise (AN).

The simplest criterion for estimating the resolution was introduced by Rayleigh in relation to problems of the theory of optical devices. According to this criterion, two identical point sources are considered to be resolved if the total response of the device for the corresponding coordinate, a parameter $l$, has two maxima. Obviously, the interval between responses, the resolution interval at which the above condition is met, coincides with the corresponding response width of the device. For the first time in relation to radio signals, the Rayleigh criterion of resolution was considered by F. Woodward [1].

The shape of the signal at the output of the receiver, which is optimal when receiving an undistorted signal against a background of white noise, at the parameter $l$ is determined by the autocorrelation function of the signal for this parameter $\rho(l)$. Therefore, the Rayleigh characteristic of the resolution coincides with the width of the main peak of the module of the autocorrelation function. Often, the resolution interval is determined by the width of the square of the autocorrelation function module. In practice, both methods of determining the resolution interval are equivalent.

As a measure of the resolution interval, the width of the square of the autocorrelation function module will be used. Quantitatively, the width of the main peak of the square of the autocorrelation function module, the resolution interval $l_p$ can be estimated in various ways, for example, by a certain level specified in a certain way, or by the width of an area-equivalent rectangle. The second method of estimation is most widely used, both due to the simplicity of definition and the unambiguity of the obtained estimates [2]. In the presence of fluctuating MN, the output signal of the receiving device is the implementation of a non-stationary random process. Therefore, the resolution interval can only be defined statistically, for example, the equivalent width of a function describing the dependence of the average output power of the receiver on the parameter $l$.

## II. WOODWARD CRITERION

When assessing a resolution, the Woodward criterion is to a certain extent conditional and makes sense only in relation to the resolution of signals of the same intensity. When distinguishing a weak signal from a strong one, it is necessary to take into account the behavior of the signal autocorrelation function $\rho(l)$ for all values of the interval

978-1-7281-8504-0/21/$31.00 ©2021 IEEE

$\Delta l$ between signals, and not only in the vicinity of the main peak of the autocorrelation function $\Delta l \leq L$, where $L$ is the total length of the signal at the parameter $l$. It becomes especially important in the presence of MN. Due to the influence of MN, the output signal of the receiving device decreases, and the relative level of signal power beyond its main maximum increases. Therefore, in the presence of MN, the mutual influence of signals increases beyond the main maximum of the autocorrelation function of the distorted signal. In addition, the Woodward criterion has another drawback: this criterion does not allow us to take into account the influence of AN on the resolution characteristics.

When using the Woodward criterion to quantify the resolution in the presence of MN, additional restrictions arise due to the challenges of this problem, such as the presence of two components in the signal distorted by MN. The response generated by the undistorted part of the signal has the same resolution width as the receiver's response to the undistorted signal.

The response width generated by the noise component can be significantly larger. In cases when the distribution of the total power of the signal distorted by MN at the output of the reception device at the parameter $l$, the amount of power of the undistorted part and the noise component of the signal has a sharp "emission" in the vicinity of the points where the signals are undistorted, the use of the Woodward criterion to assess the impact of MN on the performance of the resolution is possible. This occurs under the following assumptions about the energy characteristics of the signal at the output of the linear part of the receiver:

– the power of the undistorted part of the signal is small compared to the power of the noise component $\sigma_s^2(l)$ at the point where the undistorted part reaches its maximum;

– the function $\sigma_s^2(l)$ is sufficiently smooth and convex on the interval which is not smaller than the equivalent width of this function.

In cases where one or both of the stated conditions are not met, applying the Woodward criterion may result in errors.

Although the above conditions significantly limit the possibility of applying the Woodward criterion when analyzing the effect of MN on signal resolution conditions, this criterion is convenient due to its simplicity. In cases where the stated conditions are met, it allows us to obtain simple relations that are suitable for engineering estimates. We will consider and analyze the effect of MN on resolution of radio systems according to the Woodward criterion.

## III. Estimating the Impact of MN on Resolution by the Woodward Criterion

Let us consider the influence of quasi-deterministic and fluctuating MN on delay resolution $\tau$ and frequency resolution $\omega$ of radio signal processing systems based on the Woodward criterion.

In the absence of MN, the intervals for delay resolution and frequency resolution can be found based on the expressions:

$$\tau_{r.0} = \int_{-\infty}^{\infty} \left|\dot{\rho}(\tau,0)\right|^2 d\tau; \qquad (1)$$

$$\omega_{r.0} = \int_{-\infty}^{\infty} \left|\dot{\rho}(0,\omega)\right|^2 d\omega, \qquad (2)$$

where $\dot{\rho}(\tau,\Omega)$ is an autocorrelation function of the signal.

Expressions (1), (2) define the resolution intervals as a rectangle width with the height equal to, equivalent in area to the function $\left|\dot{\rho}(\tau,0)\right|^2 \left(\left|\dot{\rho}(0,\omega)\right|^2\right)$.

In the presence of quasi-deterministic or fluctuating MN, it makes sense to consider only the means of ensemble values included in expressions (1), (2).

Note, that in the presence of MN, the Woodward criterion can be used only in cases when the power level of the undistorted part of the signal is small as compared to the power of the noise component of the signal at such parameter $\tau$ or $\omega$, at which the undistorted part of the signal reaches its maximum, while the function $\sigma_s^2(\tau,\omega)$ describing the power distribution of the noise component on the plane $\tau$, $\omega$, is smooth. In this case, the undistorted part of the signal can be ignored, and the resolution intervals can be defined as the equivalent width of the area occupied by the noise component. Then, by analogy with (1), (2), we write the relations for determining the resolution intervals for delay and frequency in the presence of MN:

$$\left.\begin{aligned} \tau_{r.m} &= \frac{1}{\sigma_s^2(0,0)}\int_{-\infty}^{\infty} \sigma_s^2(\tau,0)\,d\tau; \\ \omega_{r.m} &= \frac{1}{\sigma_s^2(0,0)}\int_{-\infty}^{\infty} \sigma_s^2(0,\omega)\,d\omega, \end{aligned}\right\} \qquad (3)$$

where $\sigma_s^2(\tau,\omega)$ is the variance of the noise component of the signal at the output of the linear part of the receiver.

In expressions (3), it is assumed that the function $\sigma_s^2(\tau,\omega)$ has a single maximum coinciding with the maximum of the function $\left|\dot{\rho}(\tau,\omega)\right|^2$ in both coordinates. The latter condition is obviously satisfied if the energy spectrum of the noise modulation function $\dot{M}(t)$ is symmetric with respect to zero.

Here $\dot{M}(t) = \eta(t)\exp\{i\varphi(t)\}$ is the noise modulation function (NMF), which fully characterizes the parasitic modulation of the signal; $\eta(t) = \eta_0\left[1+\xi(t)\right] \geq 0$ is a dimensionless multiplier that characterizes changes in the signal envelope caused by MN (amplitude distortion); $\eta_0$ is mathematical expectation $\eta(t)$; $\eta(t)$ is a stationary random process with zero mean $\left[1+\xi(t)\right] \geq 0$; $\varphi(t)$ are changes in the signal phase caused by MN (phase distortion).

Note that the correlation function $\dot{M}(t)$ is a real function.

For rather narrow-band MN, when there is a limit relation

$$\sigma_s^2(\tau,\Omega) = 0{,}5C^2E^2\left(\overline{\eta}^2 - \alpha_0^2\right)\left|\dot{\rho}(\tau,\Omega)\right|^2,$$

as expected, (3) transforms into (1), (2).

Here $\Omega = \omega_s - \omega_0$ is the detuning of the signal $u(t)$ undistorted by fluctuation MN, with the carrier frequency $\omega_r$, with respect to the filter setting frequency $\omega_0$; $C$ is a constant coefficient depending on the filter gain; $E$ is signal energy; $\alpha_0$ characterizes a part of the signal undistorted by fluctuation MN; $\dot{\rho}(\tau,\Omega)$ is the autocorrelation function of the signal.

A straight line at the top means averaging over the set.

We express the resolution intervals of signals distorted by MN in terms of arrival time and frequency (3) directly through the characteristics of the input signals and the characteristics of the NMF.

## IV. Frequency resolution intervals

In accordance with (3), in order to determine frequency intervals, first of all it is necessary to consider the following integral

$$I_1 = \int_{-\infty}^{\infty} \sigma_s^2(0,\omega)\, d\omega.$$

Taking into account the general expression for the variance of fluctuations of the signal distorted by the fluctuation MN, at the output of the linear part of the receiver matched with the undistorted signal, for $I_1$ we obtain

$$I_1 = \frac{C^2E^2}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_V(\Omega)\left|\dot{\rho}(0,\omega+\Omega)\right|^2 d\Omega d\omega,$$

where $G_V(\Omega)$ is energy spectrum of fluctuations of the NMF.

As, by definition,

$$\int_{-\infty}^{\infty}\left|\dot{\rho}(0,\omega+\Omega)\right|^2 d\omega = \int_{-\infty}^{\infty}\left|\dot{\rho}(0,x)\right|^2 dx = \omega_{r.0},$$

then, provided that MN does not change the average signal power is $\left(\overline{\eta^2(t)}=1\right)$, taking into account the expression [1]

$$G_V(0) = \frac{2\pi\left(1-\alpha_0^2\right)}{\Delta\Omega_m},$$

where $\Delta\Omega_m$ is the equivalent width of the NMF fluctuation spectrum, we will get

$$I_1 = \frac{C^2E^2}{4\pi}\left(1-\alpha_0^2\right)\omega_{r.0}.$$

In this case, for the frequency resolution interval of signals in the presence of MN $\omega_{r.m}$, a very simple expression is obtained

$$\omega_{r.m} = \frac{\omega_{r.0}\left(1-\alpha_0^2\right)}{2\delta_1^2(0,0)}, \tag{4}$$

where

$$\delta_1^2(\tau,\Omega) = \frac{\sigma_s^2(\tau,\Omega)}{C^2E^2} \tag{5}$$

is normalized variance of the noise component of the signal, defined by the expression

$$\delta_1^2(\tau,\Omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_V(\omega)\left|\dot{\rho}(\tau,\Omega+\omega)\right|^2 d\omega.$$

The value $\delta_1^2(0,0)$ can be expressed simply by the characteristics of the NMF and the signal envelope at slow and fast MN. In the case of slow MN, we get

$$\delta_1^2(0,0) \simeq \mathrm{Re}\Big\{a_0(0,0)\dot{B}_V(0) + a_1(0,0)\dot{B}'_V(0) + \frac{1}{2}a_2(0,0)\dot{B}''_V(0)\Big\}. \tag{6}$$

where $\dot{B}'_V(\tau)$ are derivatives of the correlation function of the NMF fluctuations.

Assuming the correlation function of NMF fluctuations $\dot{B}'_V(\tau)$ to be a real function and the distortions of the amplitude and phase of the received signal are independent, with the accepted normalization of the total power of the signal distorted by MN $\left(\overline{\left|\dot{M}(t)\right|^2}=1\right)$, we have

$$\dot{B}_V(0) = 1-\alpha_0^2;\quad \dot{B}''_V(0) = -\left(1-\alpha_0^2\right)\overline{\Delta\Omega_m^2};\quad \omega_{r.m} \simeq \frac{\omega_{r.0}}{1-\overline{\Delta\Omega_m^2 t^2}}, \tag{7}$$

where $\overline{\Delta\Omega_m^2} = \int_{-\infty}^{\infty}\Omega^2 G_V(\Omega)d\Omega \Big/ \int_{-\infty}^{\infty} G_V(\Omega)d\Omega$ is the RMS width of the spectrum of the NMF.

In the case of fast MN in the expression

$$\delta_1^2(0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_V(\Omega)\left|\dot{\rho}(0,\Omega)\right|^2 d\Omega, \tag{8}$$

included in the general formula for $\omega_{r.m}$ (4), the function $G_V(\Omega)$ changes much more slowly than $\left|\dot{\rho}(0,\Omega)\right|^2$.

Expanding $G_V(\Omega)$ into a Taylor series in the vicinity of the point $\Omega = 0$ and using only three terms of the expansion, we get the case when $G_V(\Omega) = G_V(-\Omega)\left[G'_V(0)=0\right]$,

$$\omega_{r.m} \simeq \frac{\Delta\Omega_m}{1+\frac{1}{2}\overline{\omega_{r.0}^2}G''_{V.0}(0)}, \tag{9}$$

where $\overline{\omega_{r.0}^2} = \int_{-\infty}^{\infty}\omega^2\left|\dot{\rho}(0,\omega)\right|^2 d\omega \Big/ \int_{-\infty}^{\infty}\left|\dot{\rho}(0,\omega)\right|^2 d\omega$; $G_{V.0}(\Omega) = \left(G_V(0)\right)^{-1} G_V(\Omega)$.

As the width of the NMF spectrum $\Delta\Omega_m$ increases, the function $G''_{V.0}(0)$ monotonically decreases if $G_V(\Omega)$ is a smooth convex function. In this case, as can be seen from (9), the resolution interval $\omega_{r.m}$ tends to $\Delta\Omega_м$, that is, the frequency resolution is determined by the width of the NMF spectrum.

Note that expression (9) allows us to generally estimate the limits of validity of such a conclusion, that is to set the value $\Delta\Omega_m$ at which $\omega_{r.m} \simeq \Delta\Omega_m$.

## V. Intervals of time resolution

When defining time resolution intervals in accordance with (3) and the expression

$$\sigma_s^2(\tau,\Omega) = \frac{C^2E^2}{4\pi}\int_{-\infty}^{\infty} G_V(\omega)\left|\dot{\rho}(\tau,\Omega+\omega)\right|^2 d\omega = C^2E^2\delta_1^2(\tau,\Omega),$$

where $\delta_1^2(\tau,\Omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}G_V(\omega)\left|\dot{\rho}(\tau,\Omega+\omega)\right|^2 d\omega$ is normalized variance, the following integral must be considered

$$I_2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G_V(\Omega)\left|\dot{\rho}(\tau,\Omega)\right|^2 d\Omega d\omega.$$

Representing $\left|\dot{\rho}(\tau,\Omega)\right|^2$ as a double integral of the complex envelopes of the $t_1$ and $t_2$ signals, after sequentially integrating over $\tau$ and $\Omega$, and then after replacing the variables $t_1=t,\ t_1-t_2=x$, integrating over $t$, we get the expression for $I_2$ in the form of a single integral:

$$I_2=\int_{-\infty}^{\infty}B_V(x)\left|\dot{\rho}(x,0)\right|^2 dx. \tag{10}$$

Substituting (10) in (3) gives the desired expression for the time resolution integral in the presence of MN

$$\tau_{r.m}=\int_{-\infty}^{\infty}B_V(\tau)\left|\dot{\rho}(\tau,0)\right|^2 d\tau \Big/ 2\delta_1^2(0,0), \tag{11}$$

where $\delta_1^2(0,0)$ is determined by (5).

We will estimate the effect of MN on time resolution intervals for signals with $\left|\dot{\rho}(\tau,\Omega)\right|=p(\tau)r(\Omega)$, where $p(\tau)$ is the probability density of a random variable $\tau$; $r(\Omega)$ is the correlation coefficient $\Omega$. For this type of signal $I_2=2\tau_{r.0}\delta_1^2(0,0)$. Substituting the above expression in (3), it can be seen that $\tau_{р.м}=\tau_{р.0}$, which means that MN does not affect time resolution intervals when using signals whose autocorrelation function can be represented as $\left|\dot{\rho}(\tau,\Omega)\right|=p(\tau)r(\Omega)$. Such autocorrelation function is provided, in particular, by bell pulse signals with a constant pulse-modulated frequency, as well as pulse signals with a rectangular envelope and a constant pulse-modulated frequency, signals with noise modulation and phase-code manipulation at large values of the product of the spectrum width by the signal duration [1, 3, 4].

For slow MN, the correlation function $B_V(\tau)$ included in (11) can be represented by a Taylor series in the vicinity of the point $\tau=0$ and restricted to the first terms of the expansion, since it changes more slowly than $\left|\dot{\rho}(\tau,0)\right|^2$. Then, taking into account (7), (8), we get

$$\tau_{r.m}\simeq\tau_{r.0}\frac{1-\frac{1}{2}\overline{\Delta\Omega_m^2\tau_{r.0}^2}}{1-\overline{\Delta\Omega_m^2 t^2}}, \tag{12}$$

where

$$\overline{\tau_{r.0}^2}=\int_{-\infty}^{\infty}\tau^2\left|\dot{\rho}(\tau,0)\right|^2 d\tau\Big/\int_{-\infty}^{\infty}\left|\dot{\rho}(\tau,0)\right|^2 d\tau=$$
$$=\frac{1}{\tau_{r.0}}\int_{-\infty}^{\infty}\tau^2\left|\dot{\rho}(\tau,0)\right|^2 d\tau.$$

In the case of fast MN, by expanding the signal autocorrelation function $\dot{\rho}(\tau,0)$ into a Taylor series, which changes much more slowly than $B_V(\tau)$, from (11) taking into account (9), we obtain the following approximate expression for the resolution interval:

$$\tau_{r.m}\simeq\frac{2\pi}{\omega_{r.0}}\frac{1-\overline{\tau_{k.V}^2}\,\overline{\Omega^2}}{1-\frac{1}{2}\overline{\omega_{r.0}^2\tau_{k.V}^2}},$$

where

$$\overline{\tau_{k.V}^2}=\int_{-\infty}^{\infty}\tau^2 B_V(\tau)d\tau\Big/\int_{-\infty}^{\infty}B_V(\tau)d\tau$$

is RMS correlation interval of the NMF; $\overline{\Omega^2}$ is RMS width of the signal spectrum.

As the NMF correlation interval decreases and the width of its energy spectrum increases, the value $\overline{\tau_{k.V}^2}$ monotonically tends to zero. In this case, when $\overline{\tau_{k.V}^2}\to 0$, the limit value of the resolution interval is determined by the ratio

$$\tau_{r.m}\simeq\frac{2\pi}{\omega_{r.0}}. \tag{13}$$

According to (2)

$$\omega_{r.0}=\int_{-\infty}^{\infty}\left|\dot{\rho}(0,\omega)\right|^2 d\omega=\frac{1}{4E^2}\iiint_{-\infty}^{\infty}U^2(t_1)U^2(t_2)\times$$
$$\times\exp\{j\omega(t_1-t_2)\}dt_1dt_2d\omega, \tag{14}$$

where $U(t)$ is envelope of the signal.

Taking into account, that

$$\int_{-\infty}^{\infty}\exp\{j\omega(t_1-t_2)\}d\omega=2\pi\delta(t_1-t_2),$$

from (13) and (14) we will get

$$\tau_{r.m}\simeq\frac{4E^2}{\int_{-\infty}^{\infty}U^4(t)dt}.$$

For signals with rectangular envelopes with duration $T$: $2E=U^2T$ and $\tau_{r.m}=T$. For a bell-shaped signal envelope $\left[U(t)=\exp\{-\pi t^2/T^2\}\right]$, the limit value of the resolution interval $\tau_{r.m}$ is also determined by the duration $T$ of the signal.

Thus, for very wide-band MN, the time resolution interval is determined only by the signal envelope and does not depend on its phase structure. For signals with a rectangular and bell-shaped envelope, it is equal to the equivalent signal duration.

## VI. Examples of Calculating Resolution Intervals under the Influence of Multiplicative Noise

The resolution intervals will be calculated in the following order. First, we will determine the resolution intervals for slow and fast MN in relation to signals defined only by the shapes of their envelopes and energy spectra, and then, based on the general expressions (6) and (11), we will calculate the frequency and arrival time resolution intervals for specific probing signals.

Especially suitable for use in the estimation of multiplicative noise under conditions of resolution signals with intra-pulse angular modulation.

We will express the parameters $\omega_{r.0}$, $\overline{\omega_{r.0}^2}$, $\tau_{r.0}$, $\overline{\tau_{r.0}^2}$ characterizing the properties of undistorted signals in the formulas (14), (9), (1), (12), through the envelope and energy spectrum of signals calculated by authors earlier.

Considering known expressions for the delay and frequency autocorrelation function [3, 5] for the signal parameters of interest to us, the following ratios can be obtained:

$$\omega_{r.0} = 2\pi\int_{-\infty}^{\infty} U^4(t)\,dt \Big/ \left[\int_{-\infty}^{\infty} U^2(t)\,dt\right]^2; \qquad (15)$$

$$\overline{\omega_{r.0}^2} = -2\pi\int_{-\infty}^{\infty} U^2(t)\left[U^2(t)\right]''\,dt \Big/ \left[\int_{-\infty}^{\infty} U^2(t)\,dt\right]^2; \quad (15a)$$

$$\overline{\tau_{r.0}^2} = -2\pi\int_{-\infty}^{\infty} G_s(\omega)\,G_s''(\omega)\,d\omega \Big/ \left[\int_{-\infty}^{\infty} G_s(\omega)\,d\omega\right]^2; \quad (16)$$

$$\overline{\tau_{r.0}^2} = -2\pi\int_{-\infty}^{\infty} G_s(\omega)\,G_s''(\omega)\,d\omega \Big/ \left[\int_{-\infty}^{\infty} G_s(\omega)\,d\omega\right]^2. \quad (16a)$$

From expressions (15), (15a), (16), (16a) we get: expressions (17) – with slow MN, expression (18) – with fast MN, where $\delta T = \Delta_T / T$; $\delta\omega = \Delta_\omega / \Delta\omega_s$:

$$\left.\begin{aligned}
\frac{\omega_{r.m}}{\omega_{r.0}} &\simeq \left[1 - \overline{\Delta\Omega_m^2}\,\frac{T^2}{4\pi\sqrt{2}}\,\frac{\delta T^3 + 2\sqrt{2}\delta T^2(1-\delta T) + \pi\delta T(1-\delta T)^2 + \frac{\pi\sqrt{2}}{3}(1-\delta T)^3}{1 - \delta T\left(1 - \frac{1}{\sqrt{2}}\right)}\right]^{-1},\\
\frac{\tau_{r.m}}{\tau_{r.0}} &\simeq \frac{1 - \frac{1}{2}\overline{\Delta\Omega_m^2}\,\frac{\pi}{\Delta\omega_s^2\,\delta\omega\sqrt{2}}\left[1 - \delta\omega\left(1 - \frac{1}{\sqrt{2}}\right)\right]^{-1}}{1 - \overline{\Delta\Omega_m^2}\,\frac{T^2}{4\pi\sqrt{2}}\,\frac{\delta T^3 + 2\sqrt{2}\delta T^2(1-\delta T) + \pi\delta T(1-\delta T)^2 + \frac{\pi\sqrt{2}}{3}(1-\delta T)^3}{1 - \delta T\left(1 - \frac{1}{\sqrt{2}}\right)}};
\end{aligned}\right\} \qquad (17)$$

$$\left.\begin{aligned}
\frac{\Delta\omega_{r.m}}{\Delta\Omega_m} &\simeq \left[1 + \frac{1}{2}G''_{V.0}(0)\,\frac{\pi}{T^2\delta T\left(1 - \frac{1}{2}\delta T\right)}\right]^{-1},\\
\tau_{rm}\,\omega_{r.0} &\simeq \frac{1 - \frac{1}{2}\overline{\tau_{k.V}^2}\,\frac{\Delta\omega_c^2}{2\pi}\left[\delta\omega^3 + 2\delta\omega^2(1-\delta\omega) + \frac{\pi}{2}\delta\omega(1-\delta\omega)^2 + \frac{\pi}{6}(1-\delta\omega)^3\right]}{\frac{1}{2\pi}\left[1 - \frac{1}{2}\overline{\tau_{k.V}^2}\,\frac{\pi}{T^2\delta T\left(1 - \frac{1}{2}\delta T\right)}\right]}.
\end{aligned}\right\} \qquad (18)$$

We will consider the effect of MN on time and frequency resolution intervals for specific narrow-band and broadband signals, and use the expressions (4), (3), (11) to determine these values. During calculations, we assume that the spectrum of NMF fluctuations is bell-shaped.

*Signal with constant pulse-modulated frequency and a bell-shaped envelope.* Substituting the expression $\sigma_s^2(0,\Omega)$ for a signal with a bell-shaped envelope, in (4), we obtain the following relation for the frequency resolution interval:

$$\frac{\omega_{r.m}}{\omega_{r.0}} = \sqrt{1+\xi^2}. \qquad (19)$$

*Linear frequency-modulated (LFM) signal.* When a signal envelope is bell-shaped and the expression for a bell-shaped signal with a constant pulse-modulated frequency is taken into account

$$\sigma_s^2(\tau,\Omega) = \frac{C^2E^2\left(\overline{\eta^2} - \alpha_0^2\right)}{2\sqrt{1+\xi^2}}\exp\left\{-\frac{\pi\tau^2}{T^2}\right\}\exp\left\{-\frac{\Omega^2T^2}{4\pi\left(1+\xi^2\right)}\right\},$$

omitting intermediate mathematical calculations, we will obtain

$$\tau_{r.m} = \tau_{r.0}\,\frac{\sqrt{1+Q_y^2}\,\sqrt{1+\xi^2}}{\sqrt{1+\xi^2+Q_y^2}}. \qquad (20)$$

It is obvious that the expression (20) for $Q_y^2 \gg \left(1+\xi^2\right)$ up to a constant factor transforms into (19), which is quite natural, since there is a linear relationship between the frequency and time shift in the auto-correlation function of the LFM signal.

In the case when the LFM signal has a rectangular envelope, in order to calculate the arrival time resolution interval based on expression (3), the approximate ratio for

$\sigma_s^2(0,\tau)$ can be used, which is true even when $\xi = \frac{1}{2\pi}\Delta\Omega_m T > 3$.

Then we get

$$\tau_{r.m} = \frac{1}{G_V(0)}\int_{-T}^{T} G_V\left(\frac{\tau Q_y}{T^2}\right)\left(1-\frac{|\tau|}{T}\right)d\tau. \quad (21)$$

For the case when the NMF spectrum is bell-shaped

$$G_V(\Omega) = G_V(0)\exp\left\{-\frac{\pi\Omega^2}{\Delta\Omega_m}\right\},$$

from (21) we get

$$\tau_{r.m} = 2\tau_{r.0}\xi\left[\Phi\left(\frac{\sqrt{2\pi}Q_y}{\xi}\right) - \frac{\xi}{2\pi Q_y}\left(1-\exp\left\{-\frac{\pi Q_y^2}{\xi^2}\right\}\right)\right], \quad (22)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x \exp\left\{-\frac{t^2}{2}\right\}dt.$$

When $Q_y^2 >> 1$ и $\xi/Q_y << 1$ from (22) we obtain the following approximate relation:

$$\tau_{r.m} \simeq \tau_{r.0}\xi. \quad (23)$$

In this case (23) coincides with (21).

Thus, when $Q_y^2 >> \xi \geq 3$ the effect of MN on the delay resolution interval when using LFM signals does not depend on the shape of the signal envelope and is determined only by the width of the NMF spectrum.

*Phase-shift (PS) signal with a rectangular envelope.* Formally, after substituting the expression

$$\frac{2\sigma_s^2(l\Delta t,0)}{E^2\left(\overline{\eta^2}-\alpha_0^2\right)} \simeq \begin{cases} \dfrac{2\sigma_s^2(0,0)}{E^2\left(\overline{\eta^2}-\alpha_0^2\right)}, & l=0; \\ \dfrac{N-|l|}{N^2}, & |l|\leq N, \end{cases} \quad (24)$$

in (3) and after some necessary calculations

$$\tau_{r.m} = \tau_{r.0}\left[1+\frac{1-\alpha_0^2}{2\delta_1^2(0,0)}\right], \quad (25)$$

where $2\delta_1^2(0,0)$ is determined by the expression [6].

Note that (25), the same as (24), is valid when $\xi << N$. In addition, it is assumed $N >> 1$.

Given that $\xi > 3\delta_1^2(0,0) \simeq \frac{1-\alpha_0^2}{\xi}$, we can see that (25) coincides with (23). In other words, the Woodward criterion formally indicates the same effect of MN on the time-of-arrival resolution for both LFM and PS signals.

However, the use of the Woodward criterion for analyzing the effect of MN on the resolution conditions of PS signals when $\xi << N$ results in erroneous results even in cases where the undistorted part of the signal $\alpha_0^2$ is zero.

This is because the function $\delta_1^2(\tau,0)$ has a pronounced narrow outlier at the point $\tau = 0$, and the width of the specified outlier is $\tau_{r.0}$.

Thus, in this case, even when $\alpha_0^2 = 0$, the above condition for the applicability of the Woodward criterion to estimating the impact of MN on the resolution conditions is not met.

The ratio of the noise power component of the output signal at the point $\tau = 0$ to its power at the point $\tau = \tau_{r.0}$ in this example is $N/\xi$ when $\xi > 3$, $N >> 1$. If the ratio is large, then the two signals can be resolved even when the difference between the arrival time is close to $\tau_{r.0}$. All other things being equal, the statement is the more true, the larger $N$, the more code elements are formed by PS signals.

Consequently, the Woodward criterion generally cannot be used to estimate the resolution interval for the time of arrival of PS signals in the presence of MN. The results obtained on the basis of this criterion make sense only in the limiting case, when $\xi \simeq N$ and the influence of the "outlier" function $\delta_1^2(\tau,0)$ at the point $\tau = 0$ can be ignored.

## VII. Conclusions

The effect of multiplicative noise on time and frequency resolution intervals for specific narrow-band and broadband signals is considered and analyzed. The effect of multiplicative noise on a signal with a constant pulse-modulated frequency and a bell-shaped envelope, on a signal with linear frequency modulation, and on a phase-shift signal with a rectangular envelope is analyzed. During the calculation, it was assumed that the spectrum of fluctuations in the noise modulation function has a bell-shaped shape. It was shown, that the effect of multiplicative noise on the delay resolution interval when using linearly frequency-modulated signals does not depend on the shape of the signal envelope and is determined only by the spectrum width of the noise modulation function. It was shown that the Woodward criterion cannot generally be used to estimate the resolution interval for the arrival time of phase-shift signals in the presence of multiplicative noise.

## References


[1] F. Mark Richards, *Fundamentals of Radar Signal Processing*. 2nd ed. McGraw-Hill Education, 2005.

[2] Steven B. Damelin, and Willard Miller, Jr., *The Mathematics of Signal Processing*. Cambridge University Press, 2012.

[3] Joseph Bentsman, *Introductijn to Signal Processing, Instrumentation, and Control: An Integrative Approach*. University of Illinois at Urbana-Champaign, USA, 2016.

[4] S. I. Ivanov, L. B. Liorumovich, and A. V. Medvedev, "Estimation of the parameters of the phase modulated signal in presence of the background noise using complete sufficient statistics», *in Proc. 2017 XX IEEE International Conference on Soft Computing and Measurements (SMC)*, St. Peterburg, Russia, 2017. DOI: 10.1109/SCM.2017.7970480

[5] S. Miller, *Probability and Random Processes*, 2nd ed. Academic Precc, 2012.

[6] V. M. Artyushenko, and V. I. Volovach, "Estimation of accuracy for information signal parameters measuring under simultaneous multiplicative and additive non-Gaussian noise," in *Proc. IEEE International Siberian conference on control and communications (SIBCON-2019)*, Tomsk, April 18–20, 2019. DOI: 10.1109/SIBCON.2019.8729636